eAppendix 1A: Analytic Comparison of Estimators

It is straightforward to compare \( \hat{\beta}_{CME} \) and \( \hat{\beta}_{SA} \) because their numerators are the same and the denominator of \( \hat{\beta}_{CME} \) differs from the denominator of \( \hat{\beta}_{SA} \) only by the term \(-\frac{1}{2} \sum_{j=1}^{2} \sum_{i=1}^{n} (X_{ij} - \bar{X}_i)^2\). More formally,

\[
\hat{\beta}_{CME} = \left( \hat{\beta}_{SA}^{-1} - \frac{\frac{1}{2} \sum_{j=1}^{2} \sum_{i=1}^{n} (X_{ij} - \bar{X}_i)^2}{\sum_{i=1}^{n} (X_{i.} - X_.)(Y_{i} - \bar{Y})} \right)^{-1}.
\]

Thus, we see that \( \hat{\beta}_{CME} = \hat{\beta}_{SA} \) only in the limiting case where \( \delta_1 = \delta_2 = 0 \), \( \sigma_1^2 = \sigma_2^2 = 0 \) and \( \lambda_1 = \lambda_2 = 1 \). Further, since the term \(-\frac{1}{2} \sum_{j=1}^{2} \sum_{i=1}^{n} (X_{ij} - \bar{X}_i)^2\) is never positive, we can see that \( \hat{\beta}_{SA} \) will always be (weakly) smaller in magnitude (i.e. closer to 0) than \( \hat{\beta}_{CME} \).
eAppendix 1B: Simulation Experiments

For every combination of the above parameter values, we generated 1000 datasets of size $n = 250$. To create each dataset, we first generated the unobserved true predictor (the $T_i$s) from a normal distribution with mean 0 and variance 1, with sub-zero values rounded up to 0, which yielded a distribution similar to the one in the VPA and BMI example. Next, we generated the observed measures of the true predictor (the $X_{i1}$s and $X_{i2}$s) by generating $\epsilon_{i1}$ and $\epsilon_{i2}$ from normal distributions with mean 0 and variances $\sigma_1^2$ and $\sigma_2^2$, respectively, and then using Equation (1) with the relevant values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$ to calculate $X_{i1}$ and $X_{i2}$ from $T_i$, $\epsilon_{i1}$, and $\epsilon_{i2}$. Last, we generated the outcomes (the $Y_i$s) by generating $\epsilon_i$ from a $N(0, \sigma_\epsilon^2 = 1)$ distribution and then using Equation (1) with $\alpha = 1$ and $\beta = 1$ to combine $T_i$ and $\epsilon_i$. For each dataset, the SI, SA, OWA, PCA, and CME methods were used to estimate $\beta$ and $\rho$ by applying the formulas in Section 2 to $Y_i$ and $X_{i1}$ and/or $X_{i2}$. The OWA method was implemented by estimating $\alpha$ and $\beta$ (or $\rho$) for each value of $w_1$ in a grid of values ranging from 0 to 1 in 0.05 increments, and then selecting the estimate that maximized the expression in Equation (4). For each method, the differences or the squared differences between the resulting estimates and the true value of $\beta$ (= 1) or $\rho$ (= 0.5) were then averaged over the 1000 datasets to calculate the average bias and the average MSE, respectively.
eFigure 1: Average Percent Bias of $\hat{\rho}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method. Each figure presents the average percent bias of $\hat{\rho}$, by method, for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$ and a particular combination of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$. The rows in which the figures are arranged correspond to various combinations of $\delta_1$ and $\delta_2$, and the columns in which the figures are arranged correspond to various combinations of $\lambda_1$ and $\lambda_2$. An asterisk following the name of a method refers to scenarios where the estimation method failed in at least some of the 1000 iterations, and ‘NA’ refers to scenarios where the estimation method failed in all of the 1000 iterations.
eFigure 2: Average Percent Bias of \( \hat{\rho} \) for \( \sigma_1^2 = 0 \) and \( \sigma_2^2 = 0.1 \) and Varying Values of \( \delta_1, \delta_2, \lambda_1, \) and \( \lambda_2, \) By Method.
eFigure 3: Average Percent Bias of $\hat{\rho}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 4: Average Percent Bias of $\hat{\rho}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 5: Average Percent Bias of $\hat{\rho}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$. By Method.
Figure 6: Average Percent Bias of $\hat{\rho}$ for $\sigma_1^2 = 0.5$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 7: Average Percent Mean Squared Error (MSE) of $\hat{\rho}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$. Each figure presents the average percent MSE of $\hat{\rho}$, by method, for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0$ and a particular combination of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$. The rows in which the figures are arranged correspond to various combinations of $\delta_1$ and $\delta_2$, and the columns in which the figures are arranged correspond to various combinations of $\lambda_1$ and $\lambda_2$. An asterisk following the name of a method refers to scenarios where the estimation method failed in at least some of the 1000 iterations, and ‘NA’ refers to scenarios where the estimation method failed in all of the 1000 iterations.
eFigure 8: Average Percent Mean Squared Error (MSE) of $\hat{\rho}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 9: Average Percent Mean Squared Error (MSE) of $\hat{\rho}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
Figure 10. Average Percent Mean Squared Error (MSE) of $\hat{\rho}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$. By Method.
eFigure 11: Average Percent Mean Squared Error (MSE) of $\hat{\rho}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
Figure 12: Average Percent Mean Squared Error (MSE) of $\hat{\rho}$ for $\sigma_1^2 = 0.5$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 13: Average Percent Bias of $\hat{\beta}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
Figure 14: Average Percent Bias of $\hat{\beta}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
Figure 15: Average Percent Bias of $\hat{\beta}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 16: Average Percent Bias of $\hat{\beta}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 17: Average Percent Bias of $\hat{\beta}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
Figure 18: Average Percent Bias of $\hat{\beta}$ for $\sigma_1^2 = 0.5$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$. By Method.
eFigure 19: Average Percent Mean Squared Error (MSE) of $\hat{\beta}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 20: Average Percent Mean Squared Error (MSE) of $\hat{\beta}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 21: Average Percent Mean Squared Error (MSE) of $\hat{\beta}$ for $\sigma_1^2 = 0$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 22: Average Percent Mean Squared Error (MSE) of $\hat{\beta}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.1$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 23: Average Percent Mean Squared Error (MSE) of $\hat{\beta}$ for $\sigma_1^2 = 0.1$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.
eFigure 24: Average Percent Mean Squared Error (MSE) of $\hat{\beta}$ for $\sigma_1^2 = 0.5$ and $\sigma_2^2 = 0.5$ and Varying Values of $\delta_1$, $\delta_2$, $\lambda_1$, and $\lambda_2$, By Method.